REPORT 1127

ONE-DIMENSIONAL ANALYSIS OF CHOKED-FLOW TURBINES[1]

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SUMMARY

Turbines for most applications requiring high work output per stage have one or more blade rows which are choked. This analysis indicated that the area ratios and equivalent blade speed are the controlling factors in the design and operation of such turbines. For the usual class of turbine, increasing the equivalent blade speed of a given stage makes the internal flow conditions less critical. Flow conditions within choked-flow turbines are not unusually sensitive to manufacturing errors in area ratio even near the stator choking limit. Six criteria are stated that will aid in establishing from test data of multistage turbines which blade rows are choked and which are not. The variation in internal flow conditions with operating conditions for a turbine equipped with an adjustable stator is determined for one application of stator adjustment.

INTRODUCTION

Turbines for most applications requiring high work output per stage have one or more blade rows which are choked for at least a part of the operating range. Choking is a condition which exists at a high pressure ratio and during which an increase in turbine pressure ratio with constant equivalent blade speed does not affect the equivalent weight flow of the turbine. As the pressure ratio is increased above this choking value, the flow conditions upstream of the choked throat are independent of the pressure ratio and dependent upon only the turbine geometry and the equivalent blade speed.

An analysis of the flow conditions upstream of such a choked throat is important for two reasons:

(1) The basic principles controlling the operation of choked-flow turbines should be understood in order that such principles may be effectively used in design and in order that the data from tests of such turbines may be analyzed.

(2) The comparative sensitivity of factors affecting turbine efficiency to operating conditions should be known if choked-flow turbines are to be operated effectively.

A method is described for extending the analysis to include flow having variations in entropy.

This report therefore presents a one-dimensional analysis made at the NACA Lewis laboratory during 1952 of flow within turbines having choking in one or more blade rows in order to

(1) Indicate those factors which control turbine design and operation
(2) Determine the effect of variation in operating conditions on internal flow conditions
(3) Determine the effect of manufacturing errors on internal flow conditions
(4) Provide aids for analysis of turbine test data

The following cases are analyzed:

(1) A choked stator followed by a choked rotor
(2) A choked rotor followed by a choked stator
(3) A stage having a choked stator and followed by a choked stator
(4) A turbine having a choked first stator and a choked last rotor, and followed by a choked exhaust nozzle
(5) A nonchoked stator followed by a choked rotor
(6) Two-stage turbine having one or more choked blade rows
(7) Turbine equipped with adjustable inlet stator

The one-dimensional analysis is based on isentropic flow through successive throats. The results are thus qualitative, showing trends and orders of magnitude.

SYMBOLS

The following symbols are used in this report:

A	flow area, sq ft
a	sonic velocity, $\sqrt{\gamma gRT}$, ft/sec
a'_{cr}	critical velocity relative to stator, $\sqrt{\frac{2\gamma}{\gamma+1} gRT'}$, ft/sec
a''_{cr}	critical velocity relative to rotor, $\sqrt{\frac{2\gamma}{\gamma+1} gRT''}$, ft/sec
c_p	specific heat at constant pressure, Btu/(lb)(°R)
g	mass ratio, lb/slug
J	mechanical equivalent of heat, ft-lb/Btu
k, l, m, n	constants
M_r	Mach number relative to rotor, W/a
M_s	Mach number relative to stator, V/a
p	pressure, lb/sq ft absolute
R	gas constant, ft-lb/(lb)(°R)
s	entropy, Btu/(lb)(°R)
T	temperature, °R

[1] Supersedes NACA TN 2810, "One-Dimensional Analysis of Choked-Flow Turbines" by Robert E. English and Richard H. Cavicchi, 1952.

U	blade velocity, ft/sec
V	absolute velocity of gas, ft/sec
W	relative velocity of gas, ft/sec
w	rate of weight flow of gas, lb/sec
$\overline{w}$	specific-mass-flow parameter, $\frac{\rho_2 V_2}{\rho_1' a_{cr,1}'}$
α	flow angle of absolute velocity of gas measured from tangential direction (fig. 1), deg
β	flow angle of relative velocity of gas measured from tangential direction (fig. 1), deg
γ	ratio of specific heats
δ	ratio of pressure to NACA standard sea-level pressure, $p/2116$
θ	ratio of temperature to NACA standard sea-level temperature, $T/518.4$
ρ	density, lb/cu ft
ψ	reaction factor

Subscripts:

1, 3, 5, 7, 9	axial stations ahead of or behind blade rows
2, 4, 6, 8, 10	stations at throats
c	compressor inlet conditions
cr	critical, state at speed of sound
u	tangential component (positive in direction of blade speed U)
x	axial component

Superscripts:

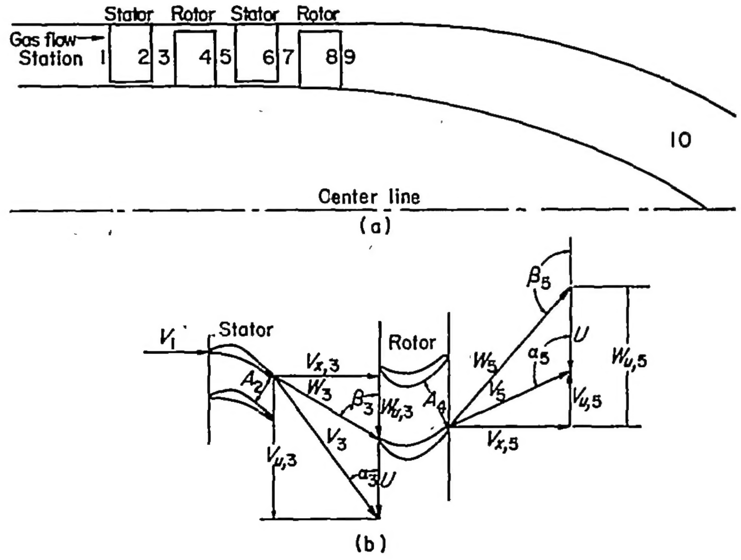
$'$	stagnation state relative to stator
$''$	stagnation state relative to rotor

GENERAL ANALYSIS

The analysis presented herein is divided into two sections, the GENERAL ANALYSIS, in which certain basic equations are presented, and the ANALYSIS AND DISCUSSION, in which various modifications of these basic equations are developed and discussed. The greater part of this investigation considers two or more choking throats; the remainder treats a turbine stage having a choked rotor but a nonchoked stator.

The basis of the analysis is an assumption of one-dimensional, isentropic flow at constant mean radius. Relative motion of the blade rows results, in general, in changes in stagnation enthalpy from one blade row to another. For such an energy extraction from a moving stream, the areas of successive choking throats differ from one another. This is in contrast with the common one-dimensional analysis of choking in successive throats (ref. 1, pp. 70–72) for which no energy is added to or extracted from the moving stream; in such a case, an assumption of constant entropy results in equal areas for the successive choking throats. Because of the assumption of one-dimensional isentropic flow at constant mean radius, the results of the investigation are qualitative.

The numerical subscripts and the axial stations within the turbine are related in figure 1. The even numbers 2, 4, 6, 8, and 10 represent stations at throats, and the odd numbers 1, 3, 5, 7, and 9 represent stations ahead of or behind blade rows.



(a) Turbine cross section showing numbered stations.
(b) Typical velocity diagram for first stage.

FIGURE 1.—Schematic illustrations showing notation used.

The stagnation state relative to any particular blade row is assumed to be constant from the entrance to the exit of each blade row; that is, for example,

$$\left.\begin{aligned} p_1'&=p_2'=p_3' \\ T_1'&=T_2'=T_3' \\ p_3''&=p_4''=p_5'' \\ T_3''&=T_4''=T_5'' \end{aligned}\right\} \qquad (1)$$

Although this assumption is not exactly satisfied in the general case, typical deviations from this assumption will not change the qualitative results of the analysis. For the assumed condition of constant radius, each of the following cases illustrates the flow conditions which may exist without a change in relative stagnation temperature across the radial element of a blade row:

(1) The flow does not shift radially across the blade row.
(2) For a stator, the flow at the entrance is isoenergetic.
(3) For a rotor, an isoenergetic flow at the entrance has a free-vortex distribution of tangential velocity.

If the stagnation values of both temperature and entropy are constant across a blade row, the stagnation pressure is also constant.

The condition of continuity of mass flow between two throats, for example, 2 and 4, can be stated.

$$\rho_2 A_2 V_2 = \rho_4 A_4 W_4 \qquad (2)$$

Equation (2) can be modified to read

$$\frac{\rho_2 V_2}{\rho_2' a_{cr,2}'} A_2 = \frac{\rho_4 W_4}{\rho_4'' a_{cr,4}''} A_4 \frac{\rho_4'' a_{cr,4}''}{\rho_2' a_{cr,2}'} \qquad (3)$$

For choking at the two throats,

$$\left.\begin{aligned}&\frac{V_2}{a'_{cr,2}}=\frac{W_4}{a''_{cr,4}}=1\\ \text{and}\\ &\frac{\rho_2}{\rho'_2}=\frac{\rho_4}{\rho''_4}=\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\end{aligned}\right\} \tag{4}$$

For a perfect gas at constant entropy,

$$\frac{\rho''_4 a''_{cr,4}}{\rho'_2 a'_{cr,2}}=\left(\frac{T''_4}{T'_2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{5}$$

Combining equations (1), (3), (4), and (5) yields

$$\frac{A_2}{A_4}=\left(\frac{T''_3}{T'_1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{6}$$

If station 4 is choked and station 2 is nonchoked, that is,

$$\frac{W_4}{a''_{cr,4}}=1 \text{ and } \frac{V_2}{a'_{cr,2}}<1$$

then

$$\frac{\rho_2 V_2}{\rho'_2 a'_{cr,2}}<\frac{\rho_4 W_4}{\rho''_4 a''_{cr,4}}=\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \tag{7}$$

For this case, equation (3) reduces to

$$\frac{\rho_2 V_2}{\rho'_2 a'_{cr,2}}=\frac{A_4}{A_2}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{T''_3}{T'_1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{8}$$

The stagnation temperature relative to a stator and relative to a rotor may be stated in terms of the velocity components in the following way:

$$c_p(T'-T)=\frac{V_x^2+V_u^2}{2gJ} \tag{9}$$

$$c_p(T''-T)=\frac{V_x^2+(V_u-U)^2}{2gJ} \tag{10}$$

With the use of

$$c_p=\frac{\gamma R}{(\gamma-1)J} \tag{11}$$

and the definition of critical velocity, equations (9) and (10) can be combined to yield

$$\frac{T''}{T'}=1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr}}\left(2\frac{V_u}{a'_{cr}}-\frac{U}{a'_{cr}}\right) \tag{12}$$

By employing

$$V_u=W_u+U$$

equation (12) can be changed to

$$\frac{T''}{T'}=1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr}}\left(2\frac{W_u}{a'_{cr}}+\frac{U}{a'_{cr}}\right) \tag{13}$$

and

$$\frac{T'}{T''}=1+\frac{\gamma-1}{\gamma+1}\frac{U}{a''_{cr}}\left(2\frac{W_u}{a''_{cr}}+\frac{U}{a''_{cr}}\right) \tag{14}$$

The equations in this section are general in the sense that they are applicable to any or all the axial stations in the turbine. Equation (6), although derived by considering conditions at stations 2 and 4, may be employed to relate temperatures and areas at any two choking stations; for example, equation (6) may be modified to read

$$\frac{A_2}{A_{10}}={\frac{T'_9}{T'_1}}^{\frac{\gamma+1}{2(\gamma-1)}} \tag{15}$$

Equations (12) to (14) are applicable to any particular axial station; for example,

$$\frac{T''_3}{T'_1}=1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr,1}}\left(2\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right) \tag{16}$$

In order to show the relation among the relative-absolute stagnation-temperature ratio T''/T', the blade speed parameter U/a'_{cr}, and the tangential velocity parameter V_u/a'_{cr}, equation (12) was used to plot figure 2. A value of 4/3 for the ratio of specific heats was employed in all the calculations for the figures presented herein. The lines of constant tangential velocity parameter V_u/a'_{cr} are parabolas passing through the point $(T''/T'=1,\ U/a'_{cr}=0)$. Along a line having a stagnation-temperature ratio T''/T' of 1,

$$\frac{U}{a'_{cr}}=2\frac{V_u}{a'_{cr}}$$

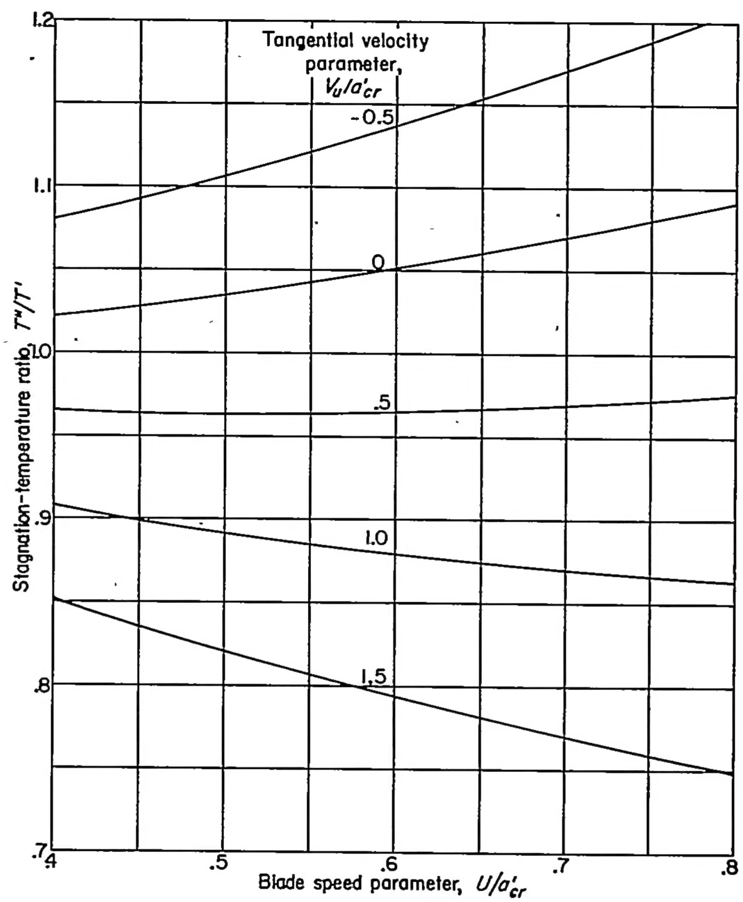


FIGURE 2.—Variation of stagnation-temperature ratio with blade speed parameter for various tangential velocity parameters (eq. (12)).

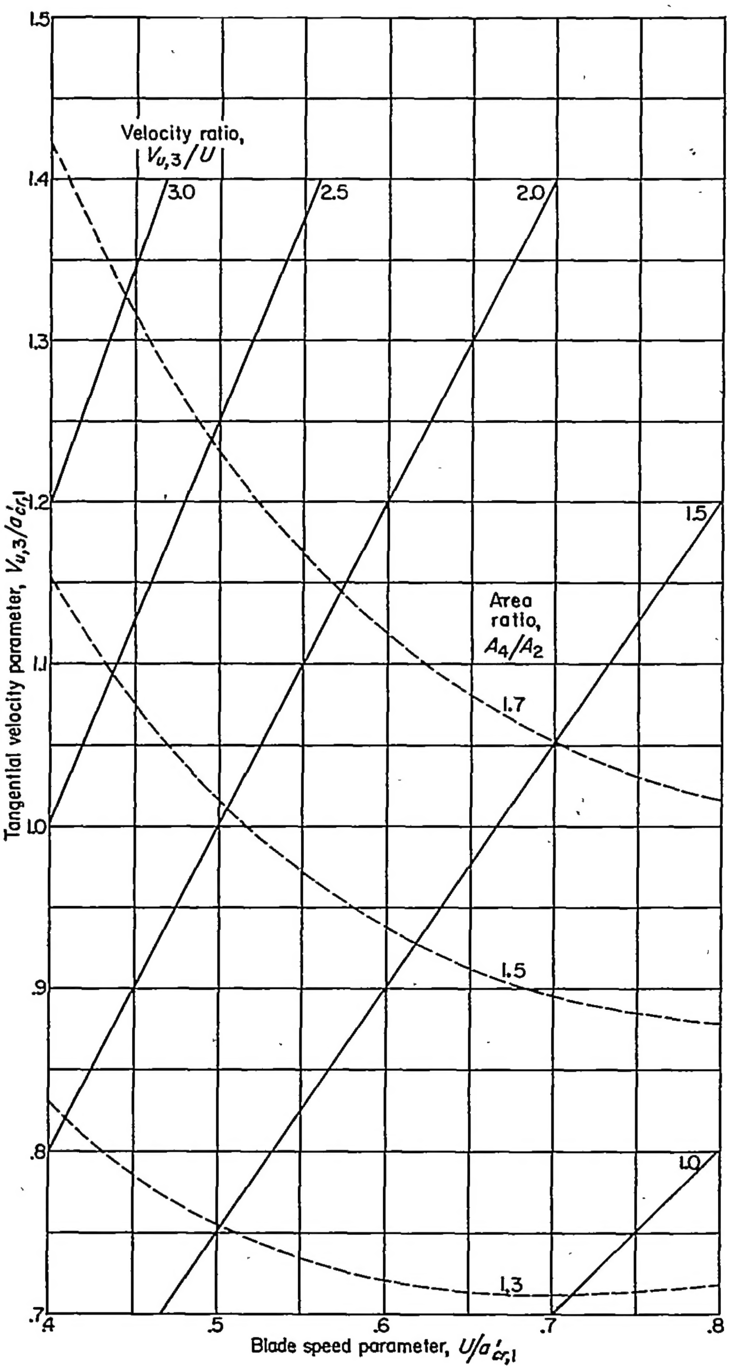


FIGURE 3.—Effect of area ratio on velocity parameters at exit of choked stator followed by choked rotor (eq. (17)).

The minimums of these parabolas occur if

$$\frac{U}{a'_{cr}}=\frac{V_u}{a'_{cr}}$$

The physical reason that the stagnation-temperature ratio T''/T' is a minimum under this condition becomes apparent upon examination of equation (10); for given values of axial velocity V_x and static temperature T, the stagnation temperature T'' is a minimum if $V_u=U$. A minimum value for the stagnation-temperature ratio is thus collateral with minimum kinetic energy relative to the rotor.

ANALYSIS AND DISCUSSION

The modifications of these basic equations, which are subsequently presented, show the effect of turbine manufacture and operation on factors influencing the turbine performance; namely, tangential velocity, entrance Mach number, turning angle, reaction, and weight flow. The following variables are postulated to be independent: area ratios, flow angles at blade-row exits, and blade speed. Blade speed is controllable during turbine operation. Area ratios and blade exit angles are established during turbine manufacture unless the exhaust nozzle or turbine stator is adjustable. In any event, the turbine geometry establishes the exit flow angle.

CHOKED STATOR FOLLOWED BY CHOKED ROTOR

Stator exit tangential velocity parameter.—Equations (6) and (12) were combined in order to determine the tangential velocity parameter $V_{u,3}/a'_{cr,1}$, a factor which considerably influences the turbine work.

$$\frac{V_{u,3}}{a'_{cr,1}}=\frac{1}{2}\left\{\frac{U}{a'_{cr,1}}+\frac{\gamma+1}{\gamma-1}\frac{a'_{cr,1}}{U}\left[1-\left(\frac{A_2}{A_4}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]\right\} \tag{17}$$

Equation (17) shows that for a given value of γ the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ may be expressed in terms of the independent variables blade speed parameter $U/a'_{cr,1}$ and area ratio A_4/A_2, as was postulated. With the turbine rotor choking, the tangential velocity parameter at the rotor entrance $V_{u,3}/a'_{cr,1}$ is independent of the over-all pressure ratio.

The lines of constant area ratio A_4/A_2 in figure 3 are a graphical representation of equation (17). The coordinates are tangential velocity parameter $V_{u,3}/a'_{cr,1}$ and blade speed parameter $U/a'_{cr,1}$; these same coordinates are used for most of the figures concerning a choked stator followed by a choked rotor. The lines of constant velocity ratio $V_{u,3}/U$ are straight lines having slopes equal to the velocity ratio $V_{u,3}/U$ and passing through the origin. The ranges of variables presented are representative of most choked-flow aircraft gas turbines.

For most of the range shown in figure 3, increasing the blade speed parameter $U/a'_{cr,1}$ along a line of constant area ratio A_4/A_2 causes a reduction in the tangential velocity parameter $V_{u,3}/a'_{cr,1}$. For a turbine being operated with a given set of inlet conditions, this means that increasing the blade speed usually reduces the tangential velocity at the stator exit; in turn, the resultant velocity at the stator exit must decrease. The range over which this trend of changing tangential velocity parameter $V_{u,3}/a'_{cr,1}$ with blade speed parameter $U/a'_{cr,1}$ is exhibited is more precisely delineated by closer examination of equation (17).

With the blade speed parameter $U/a'_{cr,1}$ and the area ratio A_4/A_2 considered to be independent, equation (17) was differentiated to yield

$$\frac{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}{\partial\left(\frac{U}{a'_{cr,1}}\right)}=1-\frac{V_{u,3}}{U} \tag{18}$$

Equation (18) shows that

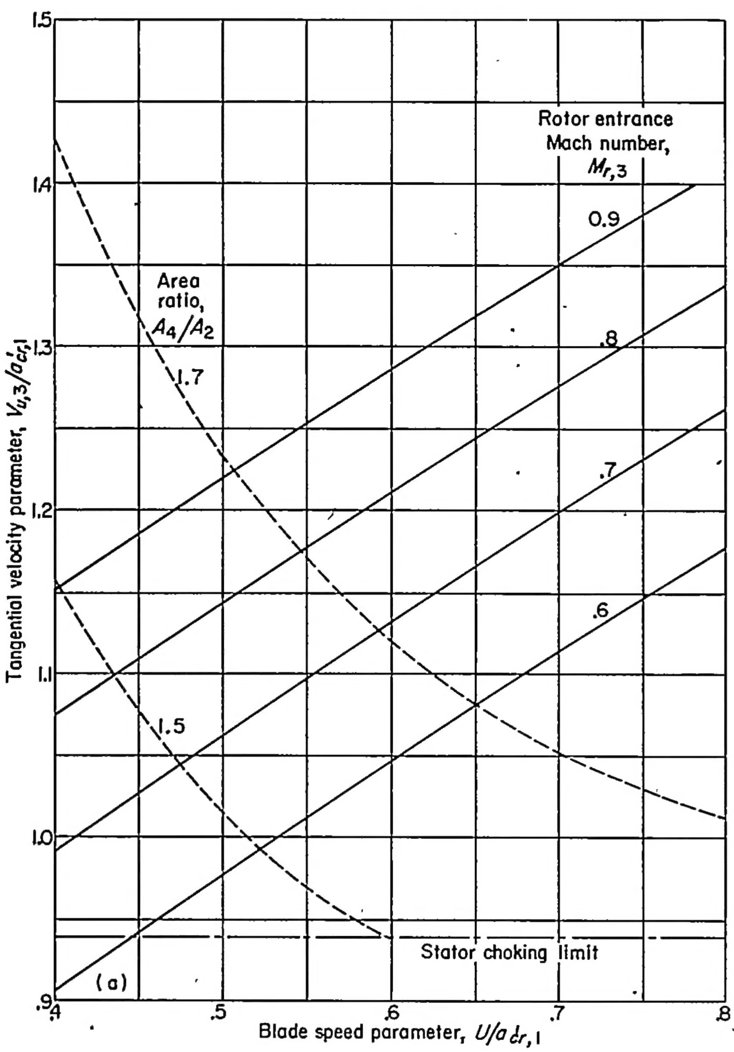
if $\frac{V_{u,3}}{U}$	$=1$	>1	<1
then $\frac{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$	$=0$	<0	>0

(19)

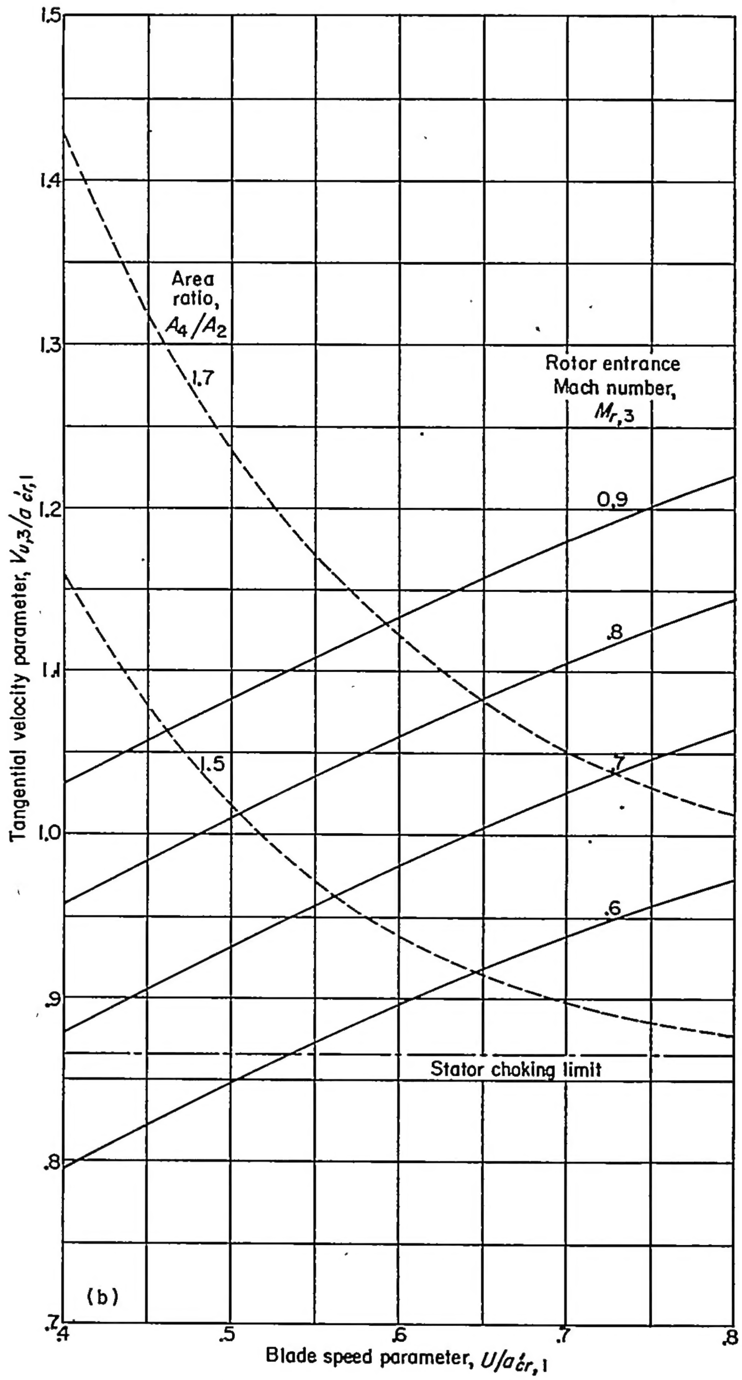
Thus, each of the lines of constant area ratio A_4/A_2 reaches a minimum where the velocity ratio $V_{u,3}/U$ is unity; this condition corresponds to axial relative flow at the rotor entrance and is typical of a turbine stage having 50 percent reaction and zero exit tangential velocity. A turbine having a velocity ratio $V_{u,3}/U$ in the neighborhood of unity therefore has essentially no change in tangential velocity parameter with blade speed parameter; whereas a turbine having a velocity ratio $V_{u,3}/U$ greater than unity experiences a decrease in tangential velocity parameter with increasing blade speed parameter.

Rotor entrance Mach number.—For either a choked or a nonchoked stator, the relative Mach number at the rotor entrance $M_{r,3}$ may be expressed

$$M_{r,3}=\sqrt{\frac{2}{\gamma+1}\,\frac{\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\tan^2\alpha_3+\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)^2}{1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3}} \tag{20}$$



(a) Stator exit flow angle, 20°.



(b) Stator exit flow angle, 30°.

FIGURE 4.—Relation among area ratio, rotor entrance Mach number, and velocity parameters at exit of choked stator followed by choked rotor (eqs. (17) and (20)).

For a choked stator followed by a choked rotor, equation (17) may be used to replace the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ in equation (20) by a function of area ratio A_4/A_2 and blade speed parameter $U/a'_{cr,1}$. For any given value of specific heat ratio γ, the rotor entrance Mach number $M_{r,3}$ in equation (20) can therefore be reduced to a function of three variables—area ratio A_4/A_2, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3. The relation among

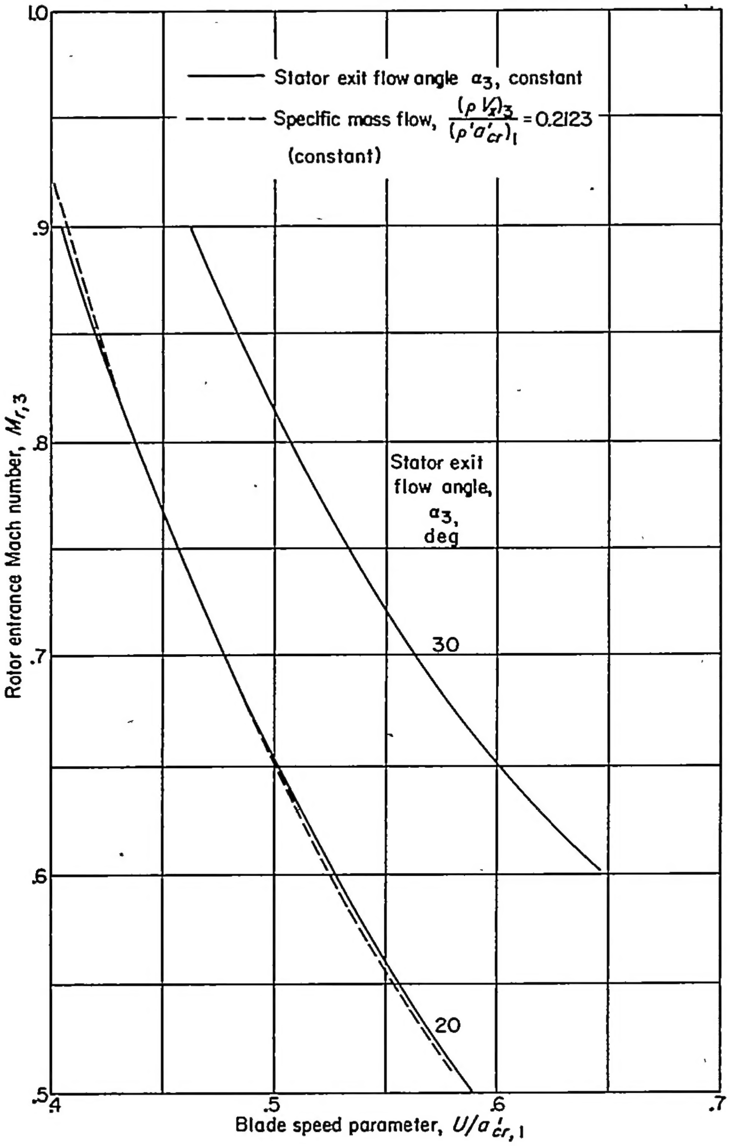


FIGURE 5.—Comparative effect of assumption of both constant stator exit flow angle and constant specific mass flow on rotor entrance Mach number at area ratio of 1.5.

these parameters is shown in figure 4. For stator choking, the velocity at the throat (station 2) is equal to the entrance value of critical velocity $a'_{cr,1}$. At the stator choking limit, the stator exit velocity V_3 is presumed to be equal to the throat velocity V_2 with the result that at the stator choking limit

$$\frac{V_{u,3}}{V_3}=\frac{V_{u,3}}{a'_{cr,1}}=\cos\alpha_3$$

For values of tangential velocity parameter $V_{u,3}/a'_{cr,1}$ above the stator choking limit, lines of constant area ratio A_4/A_2 have been reproduced from figure 3 (eq. (17)).

When figure 4 is employed to estimate the variation in rotor entrance Mach number $M_{r,3}$ with blade speed parameter $U/a'_{cr,1}$, an error is introduced by considering the stator exit flow angle α_3 to be constant. This error arises because, as the blade speed parameter is changed, the mass flow through the stator will vary for constant stator exit flow angle α_3, whereas for actual operation with a given stator the mass flow will remain constant and the stator exit flow angle α_3 will vary. The effect of this discrepancy on rotor entrance Mach number $M_{r,3}$ is shown in figure 5; the error is comparatively small. The presence of two lines of constant stator exit flow angle α_3 in figure 5 permits the deviation of the stator exit flow angle to be estimated from comparative distances separating the lines. Because this error introduces only a small discrepancy into the results, the stator exit flow angle α_3 was, for simplicity, assumed to be independent of the tangential velocity parameter $V_{u,3}/a'_{cr,1}$.

Effect of manufacturing errors.—Figure 4 shows that increasing the area ratio A_4/A_2 increases the rotor entrance Mach number $M_{r,3}$ for any given value of blade speed parameter $U/a'_{cr,1}$. In order to examine in greater detail the variation in rotor entrance Mach number $M_{r,3}$ with manufacturing errors in area ratio A_4/A_2, equation (20) was differentiated with the qualification that area ratio A_4/A_2, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 are independent variables. In formulating the desired relation, the following identity was employed:

$$\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}=-\left(\frac{1}{2M_{r,3}}\right)\left[\frac{\partial(M_{r,3}^2)}{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}\right]\left[\frac{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}{\partial\left(\frac{A_2}{A_4}\right)}\right]\left(\frac{A_2}{A_4}\right) \tag{21}$$

Evaluation of the four factors on the right side of equation (21) by means of equations (17) and (20) yields the following expression for the manufacturing-error parameter:

$$\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}=\frac{\frac{V_{u,3}}{U}\left[1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr,1}}\left(2\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\right]}{\left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]^{\frac{3}{2}}}\;\frac{\left[\sec^2\alpha_3-\frac{U}{V_{u,3}}-\frac{\gamma-1}{\gamma+1}\sec^2\alpha_3\frac{U}{a'_{cr,1}}\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\right]}{\sqrt{\frac{\gamma+1}{2}\left[\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\tan^2\alpha_3+\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)^2\right]}}. \tag{22}$$

This parameter $\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}$ is herein termed "manufacturing-error parameter." For small changes in area ratio A_4/A_2 such as might occur during turbine manufacture, the manufacturing-error parameter $\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}$ can be used to estimate the change in rotor entrance Mach number $M_{r,3}$. Equation (22) has been plotted in figure 6 in order to show the variation in manufacturing-error parameter $\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}$ with tangential velocity parameter $V_{u,3}/a'_{cr,1}$, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3.

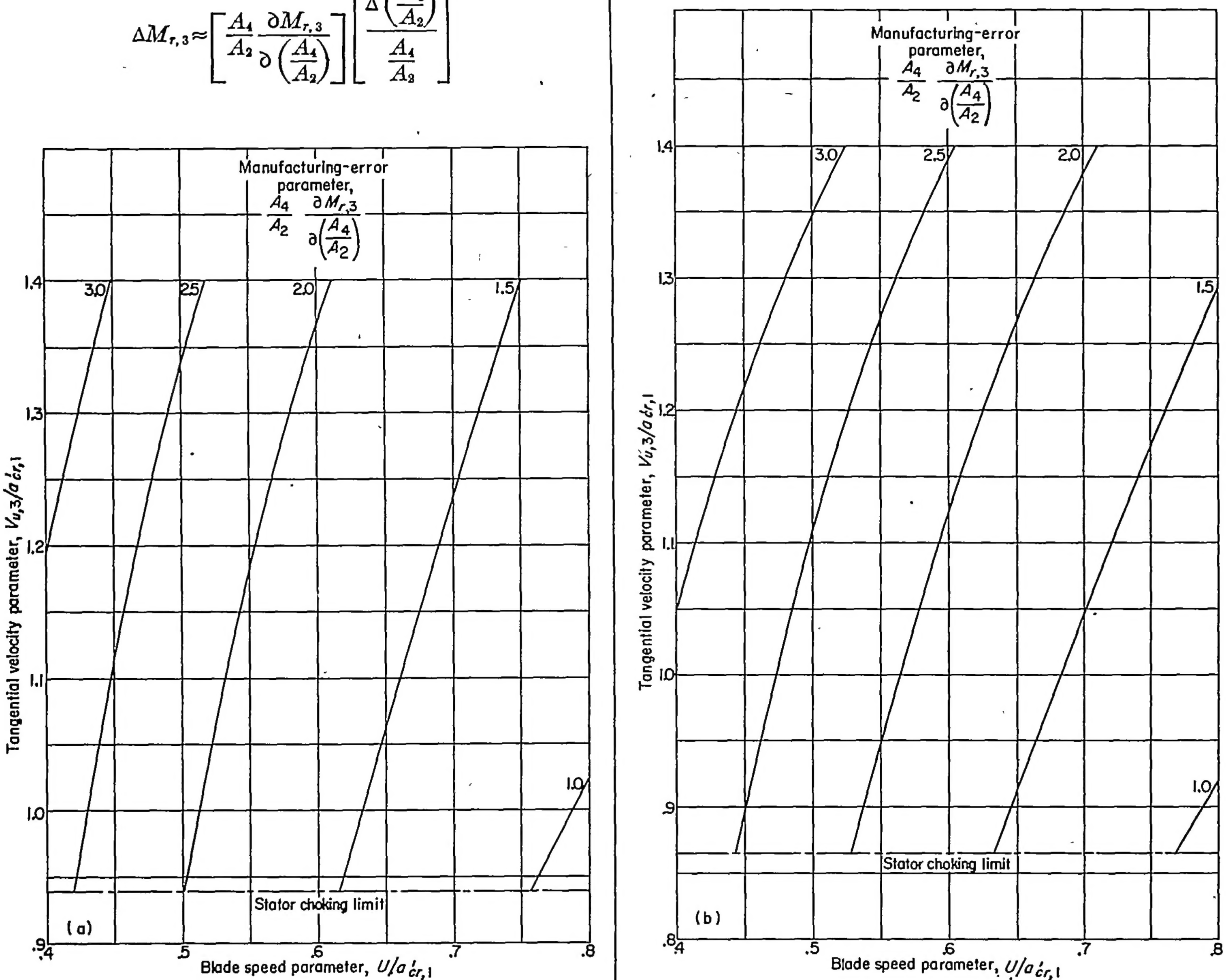
The following example illustrates the use of the manufacturing-error parameter in estimating the effect of errors in area ratio on rotor entrance Mach number for a stator exit flow angle α_3 of 20°, a tangential velocity parameter $V_{u,3}/a'_{cr,1}$ of 1.2, and a blade speed parameter $U/a'_{cr,1}$ of 0.69. Figure 6 shows that the manufacturing-error parameter $\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}$ is 1.5, and figure 4 indicates that the rotor entrance Mach number $M_{r,3}$ is 0.71. If, in the course of turbine manufacture, the rotor throat area A_4 is made 1 percent greater than the design value and the stator throat area A_2 is made 1 percent smaller than the design value, the area ratio A_4/A_2 is increased by approximately 2 percent. The increment in rotor entrance Mach number $M_{r,3}$ associated with this change in area ratio A_4/A_2 can be estimated from

$$\Delta M_{r,3}\approx\left[\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}\right]\left[\frac{\Delta\left(\frac{A_4}{A_2}\right)}{\frac{A_4}{A_2}}\right]$$

which, for this case, is 0.03. Thus, the rotor entrance Mach number $M_{r,3}$ will be increased to approximately 0.74 for operation of the turbine at the design value of blade speed parameter.

Effect of blade speed on rotor entrance conditions.—For most of the range presented in figure 4, increasing the blade speed parameter $U/a'_{cr,1}$ at constant area ratio A_4/A_2 corresponds to decreasing the rotor entrance Mach number $M_{r,3}$. In order to examine in greater detail the range of conditions for which this relation is true, equation (20) was differentiated with the qualification that area ratio A_4/A_2, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 are independent variables. The result of this differentiation is

$$\frac{\partial M_{r,3}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}=\frac{2}{M_{r,3}(\gamma+1)}\frac{V_{u,3}}{a'_{cr,1}}\left(1-\frac{V_{u,3}}{U}\right)\sec^2\alpha_3\frac{\frac{T''_3}{T'_1}}{\left(\frac{T_3}{T'_1}\right)^2} \quad (23)$$



(a) Stator exit flow angle, 20°.

(b) Stator exit flow angle, 30°.

FIGURE 6.—Effect of manufacturing errors in throat areas on rotor entrance Mach number for choked stator followed by choked rotor (eq. (22)).

where the relative-absolute temperature ratio T_3''/T_1' is stated in equation (16), and, by combination of equation (9) and the definition of critical velocity a'_{cr},

$$\frac{T_3}{T_1'}=1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2 \sec^2\alpha_3 \tag{24}$$

With the single exception of $\left(1-\frac{V_{u,3}}{U}\right)$, every factor on the right side of equation (23) is positive at the entrance to any turbine rotor. Therefore,

if $\frac{V_{u,3}}{U}$	$=1$	>1	<1
then $\frac{\partial M_{r,3}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$	$=0$	<0	>0

(25)

A reaction factor ${}_3\psi_4$ is defined

$${}_3\psi_4\equiv\frac{W_4^2-W_3^2}{W_3^2} \tag{26}$$

For choking in the rotor,

$$W_4=a''_{cr,3}$$

Equation (26) therefore reduces to

$${}_3\psi_4=\left(\frac{a''_{cr,3}}{W_3}\right)^2-1 \tag{27}$$

Because

$$\frac{dM_{r,3}}{d\left(\frac{W_3}{a''_{cr,3}}\right)}>0$$

differentiation of equation (27) and combination of this result with equation (25) yield the following:

if $\frac{V_{u,3}}{U}$	$=1$	>1	<1
then $\frac{\partial\, {}_3\psi_4}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$	$=0$	>0	<0

(28)

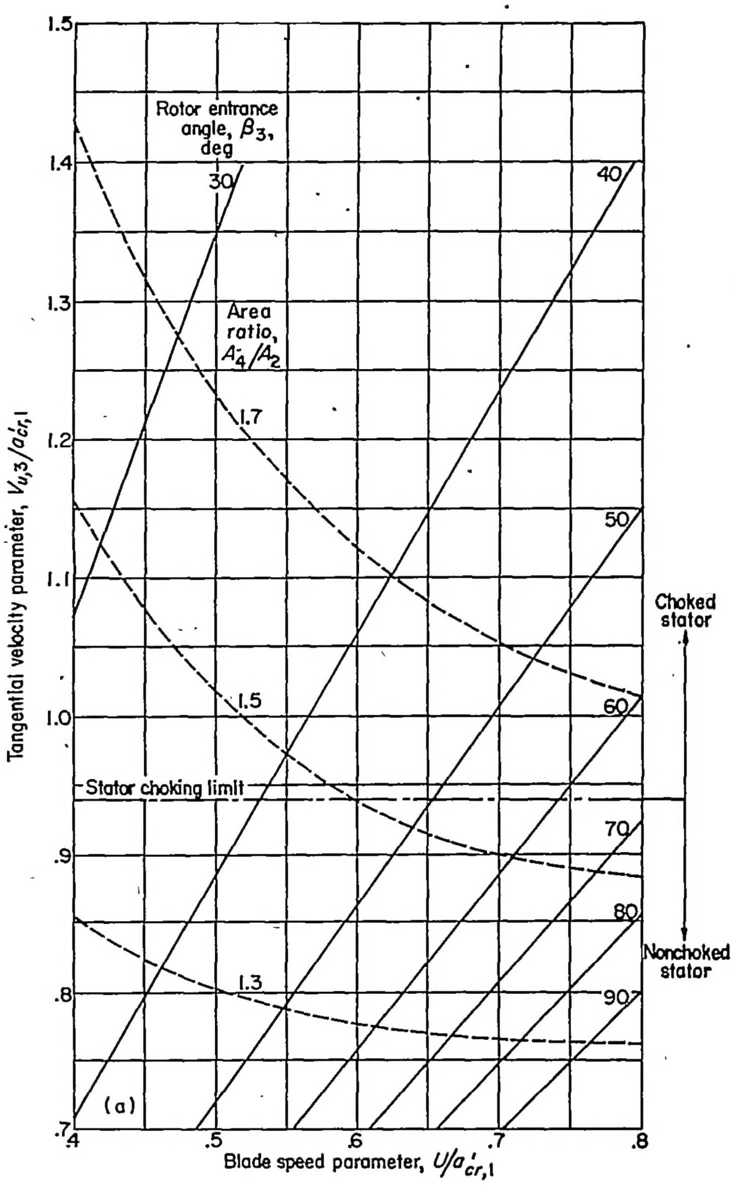
From the geometry of the vector diagram in figure 1,

$$\frac{V_{u,3}}{a'_{cr,1}}\tan\alpha_3=\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\tan\beta_3$$

or

$$\tan\beta_3=\frac{\tan\alpha_3}{1-\frac{U}{V_{u,3}}} \tag{29}$$

Equation (29) is plotted in figure 7 for two values of stator exit flow angle α_3. The lines of constant area ratio A_4/A_2 shown in the region above the stator choking limit are

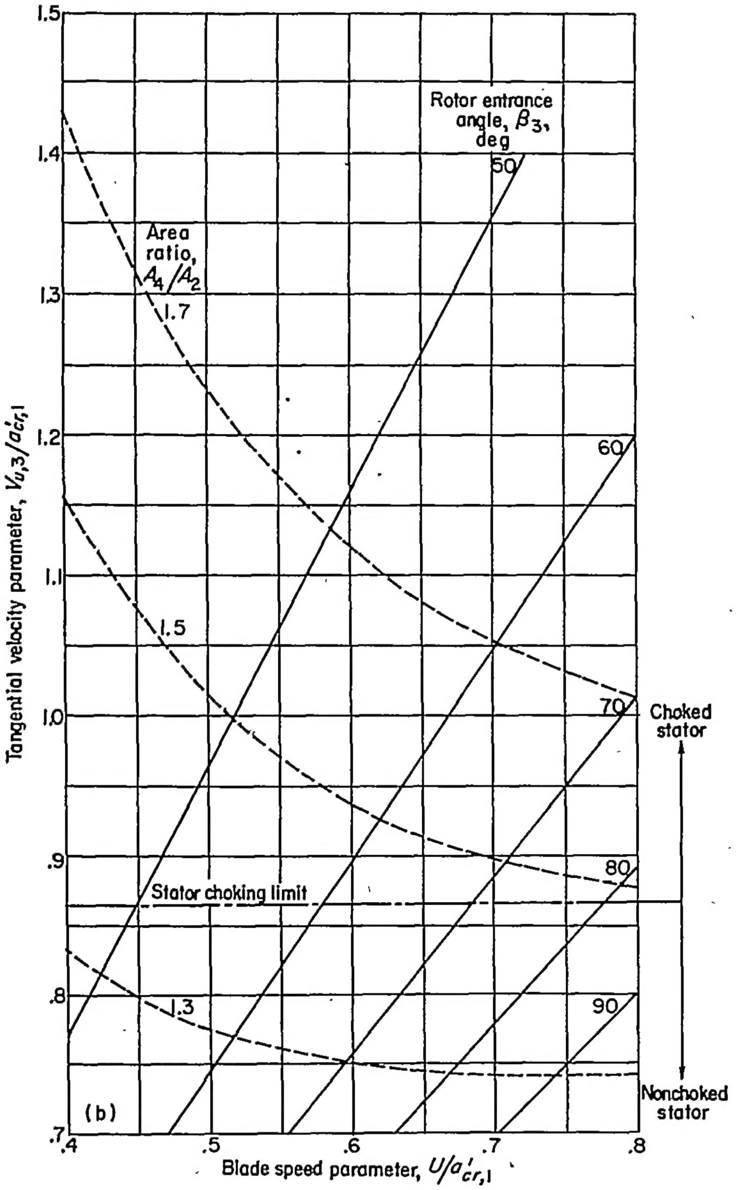


(a) Stator exit flow angle, 20°.

FIGURE 7.—Relation among area ratio, rotor entrance angle, and velocity parameters at exit of choked stator followed by choked rotor (eq. (29)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

reproduced from figure 3, whereas the extension of these lines into the region below the stator choking limit has been accomplished by means of a relation yet to be developed (eq. (50)). Along a line of constant area ratio A_4/A_2 for a given stator exit flow angle α_3, increasing the blade speed parameter $U/a'_{cr,1}$ increases the rotor entrance flow angle β_3. Increased values of rotor entrance flow angle β_3 correspond to reduced angles of attack on the rotor blades and reduced amounts of turning by the rotor.

A consideration of equations (25) and (28) and figure 7 indicates the conditions for which a turbine should be designed if that turbine is to be operated over a range of blade speed parameter $U/a'_{cr,1}$. The values of rotor entrance



(b) Stator exit flow angle, 30°.

FIGURE 7.—Concluded. Relation among area ratio, rotor entrance angle, and velocity parameters at exit of choked stator followed by choked rotor (eq. (29)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

Mach number $M_{r,3}$, reaction factor ${}_3\psi_4$, and rotor entrance flow angle β_3 are factors which are commonly considered to have critical effects on turbine efficiency. For turbine stages having $\frac{V_{u,3}}{U}>1$, small increases in blade speed parameter $U/a'_{cr,1}$ cause the magnitude of each of these factors to become more favorable. On the other hand, large increases in blade speed parameter $U/a'_{cr,1}$ will result in large negative angles of incidence and thus poor performance. Unless this is the case, the turbine should be designed for the lowest anticipated value of blade speed parameter $U/a'_{cr,1}$ in order that satisfactory rotor performance may be guaranteed for the required range of operation.

Recapitulation.—This analysis of conditions between a choked stator and a choked rotor indicates those factors which control the flow conditions between the stator and the rotor, namely, blade speed parameter $U/a'_{cr,1}$, area ratio A_4/A_2, and stator exit flow angle α_3. The importance of maintaining proper values of stator and rotor throat areas in turbine manufacture thus becomes apparent if the rotor entrance conditions are to be preserved. The sensitivity of rotor entrance Mach number $M_{r,3}$ to manufacturing errors in area ratio A_4/A_2 was determined; and rotor entrance Mach number $M_{r,3}$ was found to be only moderately sensitive to errors in area ratio A_4/A_2, even in the neighborhood of the stator choking limit. The variations in rotor entrance Mach number $M_{r,3}$, rotor entrance flow angle β_3, and reaction factor ${}_3\psi_4$ with blade speed parameter $U/a'_{cr,1}$ were determined. For the common range of design, for which $V_{u,3}>U$, each of these factors becomes more conservative as the blade speed parameter $U/a'_{cr,1}$ is increased.

CHOKED ROTOR FOLLOWED BY CHOKED STATOR

Stator entrance tangential velocity parameter.—The analysis of flow between a choked rotor and a choked stator is essentially the same as that for a choked stator followed by a choked rotor. The stator entrance tangential velocity parameter $V_{u,5}/a'_{cr,5}$ may be determined from a modification of equation (6) in combination with equation (12).

$$\frac{V_{u,5}}{a'_{cr,5}}=\frac{1}{2}\left\{\frac{U}{a'_{cr,5}}-\frac{\gamma+1}{\gamma-1}\frac{a'_{cr,5}}{U}\left[\left(\frac{A_6}{A_4}\right)^{\frac{2(\gamma-1)}{\gamma+1}}-1\right]\right\} \tag{30}$$

With the blade speed parameter $U/a'_{cr,5}$ and the area ratio A_6/A_4 considered independent variables, equation (30) may be differentiated to yield

$$\frac{\partial\left(\frac{V_{u,5}}{a'_{cr,5}}\right)}{\partial\left(\frac{U}{a'_{cr,5}}\right)}=1-\frac{V_{u,5}}{U} \tag{31}$$

Because, almost without variation,

$$\frac{V_{u,5}}{U}<1 \quad \text{then as a result} \quad \frac{\partial\left(\frac{V_{u,5}}{a'_{cr,5}}\right)}{\partial\left(\frac{U}{a'_{cr,5}}\right)}>0 \tag{32}$$

Rotor exit tangential velocity parameter.—In the same way,

$$\frac{W_{u,5}}{a''_{cr,4}}=-\frac{1}{2}\left\{\frac{U}{a''_{cr,4}}+\frac{\gamma+1}{\gamma-1}\frac{a''_{cr,4}}{U}\left[1-\left(\frac{A_4}{A_6}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]\right\} \tag{33}$$

and

$$\frac{\partial\left(\frac{W_{u,5}}{a''_{cr,4}}\right)}{\partial\left(\frac{U}{a''_{cr,4}}\right)}=\frac{-W_{u,5}}{U}-1 \tag{34}$$

Equation (34) shows that

if $\frac{-W_{u,5}}{U}$	$=1$	>1	<1
then $\frac{\partial\left(\frac{W_{u,5}}{a''_{cr,4}}\right)}{\partial\left(\frac{U}{a''_{cr,4}}\right)}$	$=0$	>0	<0

(35)

Figure 3 can be interpreted as a graphical representation of equation (33) if

$$\frac{V_{u,3}}{a'_{cr,1}} \text{ is replaced by } \frac{-W_{u,5}}{a''_{cr,4}}$$

$$\frac{U}{a'_{cr,1}} \text{ is replaced by } \frac{U}{a''_{cr,4}}$$

$$\frac{A_4}{A_2} \text{ is replaced by } \frac{A_6}{A_4}$$

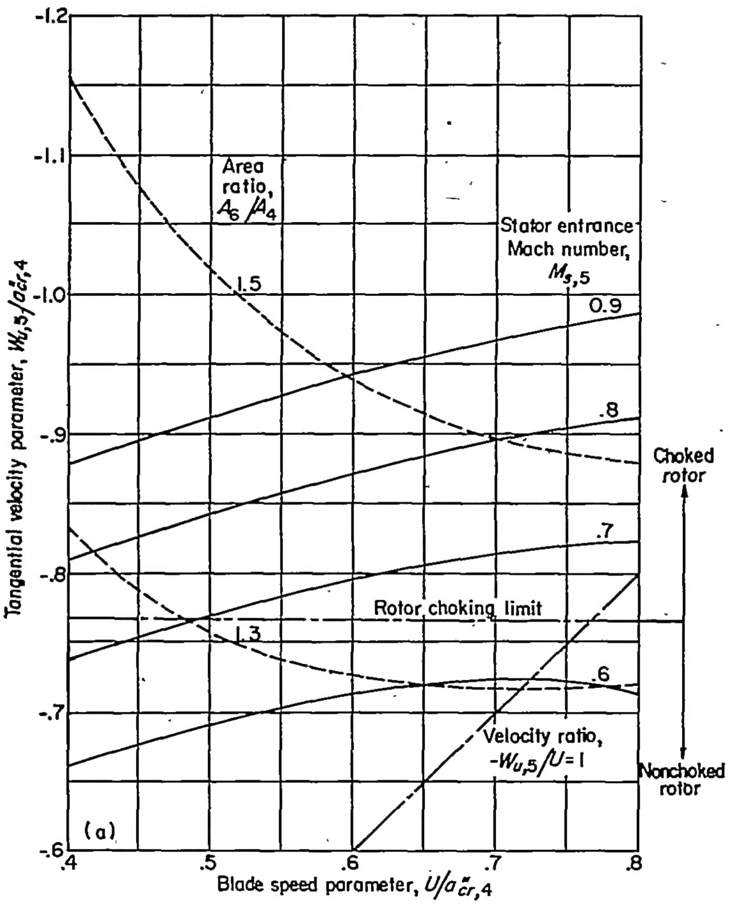
$$\frac{V_{u,3}}{U} \text{ is replaced by } \frac{-W_{u,5}}{U}$$

Stator entrance Mach number.—The absolute Mach number at the stator entrance $M_{s,5}$ may be expressed in a form similar to equation (20)

$$M_{s,5}=\sqrt{\frac{2}{\gamma+1}\,\frac{\left(\frac{W_{u,5}}{a''_{cr,4}}\right)^2\tan^2\beta_5+\left(\frac{W_{u,5}}{a''_{cr,4}}+\frac{U}{a''_{cr,4}}\right)^2}{1-\frac{\gamma-1}{\gamma+1}\left(\frac{W_{u,5}}{a''_{cr,4}}\right)^2\sec^2\beta_5}} \tag{36}$$

Equation (36) may be revised to read

$$\frac{U}{a''_{cr,4}}=\pm\sqrt{\frac{\gamma+1}{2}M_{s,5}^2\left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{W_{u,5}}{a''_{cr,4}}\right)^2\sec^2\beta_5\right]-\left(\frac{W_{u,5}}{a''_{cr,4}}\right)^2\tan^2\beta_5}-\frac{W_{u,5}}{a''_{cr,4}} \tag{37}$$



(a) Rotor exit flow angle, 140°.

FIGURE 8.—Relation among area ratio, stator entrance Mach number, and velocity parameters at exit of choked rotor followed by choked stator (eqs. (36) and (37)). For lines of constant area ratio, equation (33) applies above rotor choking limit and equation (50) applies below rotor choking limit.

The radical in equation (37) represents the tangential velocity parameter $V_{u,5}/a''_{cr,4}$. For the conventional range of design, the negative sign should be used.

In figure 8, equation (37) is plotted for values of rotor exit flow angle β_5 of 140° and 150°. Lines of constant area ratio A_6/A_4 were plotted by using equation (33). For most of the range of these maps, the stator entrance Mach number $M_{s,5}$ decreases as the blade speed parameter $U/a''_{cr,4}$ is increased along a line of constant area ratio A_6/A_4. For a velocity ratio $-W_{u,5}/U$ of unity, each line of constant stator entrance Mach number $M_{s,5}$ reaches a maximum and each line of constant area ratio A_6/A_4 reaches a minimum.

With the blade speed parameter $U/a''_{cr,4}$, the area ratio A_6/A_4, and the rotor exit flow angle β_5 considered to be independent variables, equation (36) was differentiated to yield

$$\frac{\partial M_{s,5}}{\partial\left(\frac{U}{a''_{cr,4}}\right)}=\frac{2}{M_{s,5}(\gamma+1)}\,\frac{W_{u,5}}{a''_{cr,4}}\left(\frac{-W_{u,5}}{U}-1\right)\sec^2\beta_5\,\frac{\frac{T''_5}{T'_5}}{\left(\frac{T_5}{T'_5}\right)^2} \tag{38}$$

The similarity between equations (23) and (38) is immediately apparent. Because for any practical turbine

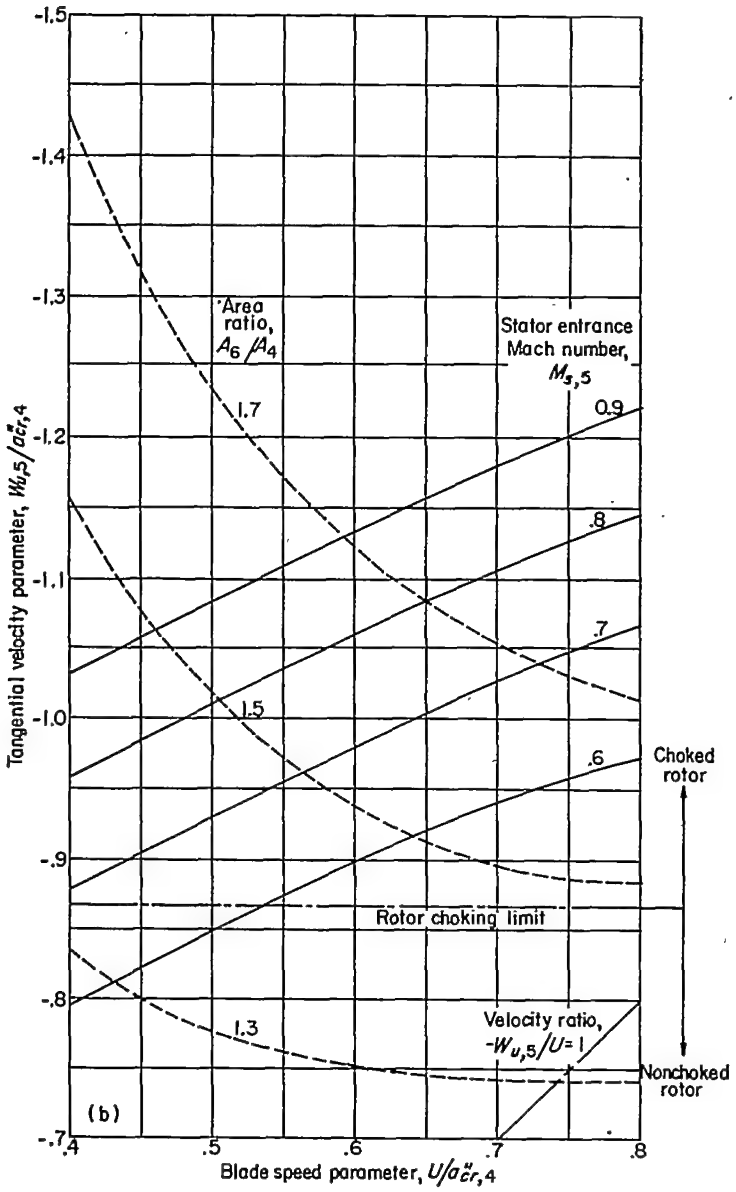
$$\frac{W_{u,5}}{a''_{cr,4}}<0$$

an equation similar to equation (25) may be written.

If $\frac{-W_{u,5}}{U}$	$=1$	>1	<1
then $\frac{\partial M_{s,5}}{\partial\left(\frac{U}{a''_{cr,4}}\right)}$	$=0$	<0	>0

(39)

In like manner, an equation similar to equation (28) may be written.



(b) Rotor exit flow angle, 150°.

FIGURE 8.—Concluded. Relation among area ratio, stator entrance Mach number, and velocity parameters at exit of choked rotor followed by choked stator (eqs. (36) and (37)). For lines of constant area ratio, equation (33) applies above rotor choking limit and equation (50) applies below rotor choking limit.

If $\frac{-W_{u,5}}{U}$	$=1$	>1	<1
then $\frac{\partial_6\psi_6}{\partial\left(\frac{U}{a'_{cr,4}}\right)}$	$=0$	>0	<0

(40)

Recapitulation.—For a choked rotor followed by a choked stator the flow conditions between the rotor and the stator have essentially the same characteristics as the flow behind a choked stator which is followed by a choked rotor. The principal variables controlling the flow conditions are the blade speed parameter $U/a''_{cr,4}$, the area ratio A_6/A_4, and the rotor exit flow angle β_5. As for a choked stator followed by a choked rotor, the flow conditions at the entrance to a choked stator which follows a choked rotor are generally improved by increasing the blade speed parameter $U/a''_{cr,4}$.

STAGE HAVING CHOKED STATOR AND FOLLOWED BY CHOKED STATOR

For choking at stations 2 and 6 (see fig. 1), equation (6) may be modified to read

$$\frac{T'_1}{T'_5}=\left(\frac{A_6}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \tag{41}$$

Equation (41) shows that for a stage having a choked stator and followed by a choked stator the stage temperature ratio T'_1/T'_5 depends upon only the area ratio A_6/A_2 and is thus independent of the blade speed parameter. If either of the stators is not choking, however, the stage temperature ratio T'_1/T'_5 is a function of blade speed parameter as well as area ratio. The significance of area ratio A_6/A_2 in determining the stage temperature ratio T'_1/T'_5 emphasizes the importance of maintaining the proper throat areas during turbine manufacture.

Differentiation of equation (41) gives

$$\frac{d\left(\frac{T'_1}{T'_5}\right)}{\frac{T'_1}{T'_5}}=\frac{2(\gamma-1)}{\gamma+1}\,\frac{d\left(\frac{A_6}{A_2}\right)}{\frac{A_6}{A_2}} \tag{42}$$

The fractional error in stage temperature ratio T'_1/T'_5 is thus about 0.3 the fractional error in area ratio A_6/A_2. For a typical stage temperature ratio T'_1/T'_5 of 1.2, the fractional error in turbine equivalent work is then about 1.5 the fractional error in area ratio A_6/A_2 for a stage having a choked stator and followed by a choked stator.

TURBINE HAVING CHOKED FIRST STATOR AND CHOKED LAST ROTOR AND FOLLOWED BY CHOKED EXHAUST NOZZLE

For choking at stations 2, 8, and 10 (see fig. 1), equation (6) can be rewritten

$$\frac{T'_1}{T''_8}=\left(\frac{A_8}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \tag{43}$$

and

$$\frac{T'_1}{T'_9}=\left(\frac{A_{10}}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \tag{44}$$

Rewriting equation (12) results in

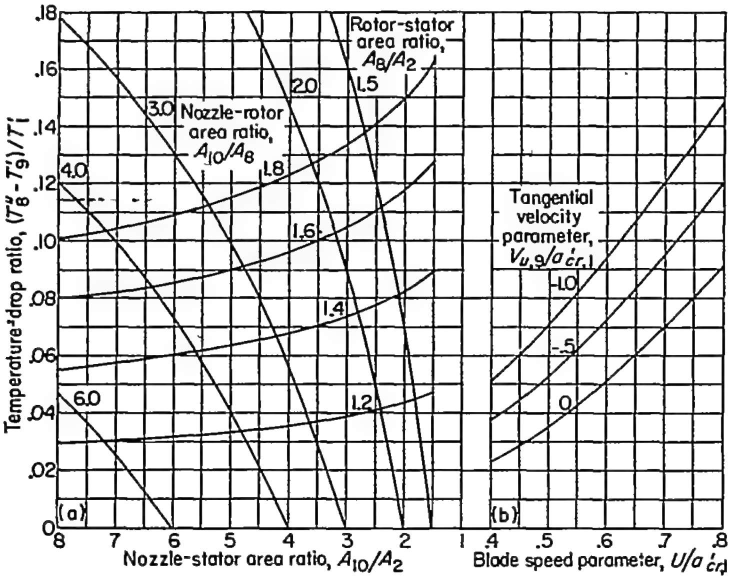
$$\frac{T''_8}{T'_9}=1-\frac{\gamma-1}{\gamma+1}\,\frac{U}{a'_{cr,9}}\left(2\,\frac{V_{u,9}}{a'_{cr,9}}-\frac{U}{a'_{cr,9}}\right) \tag{45}$$

Combining equations (43), (44), and (45) with the definition of critical velocity a'_{cr} yields

$$\frac{V_{u,9}}{a'_{cr,1}}=\frac{1}{2}\left[\frac{U}{a'_{cr,1}}-\frac{\gamma+1}{\gamma-1}\,\frac{a'_{cr,1}}{U}\left(\frac{T''_8-T'_9}{T'_1}\right)\right] \tag{46}$$

and

$$\frac{T''_8-T'_9}{T'_1}=\left(\frac{A_2}{A_8}\right)^{\frac{2(\gamma-1)}{\gamma+1}}-\left(\frac{A_2}{A_{10}}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \tag{47}$$



(a) Relation among temperature-drop ratio, nozzle-stator area ratio, rotor-stator area ratio, and nozzle-rotor area ratio (eq. (47)).
(b) Relation among temperature-drop ratio, blade speed parameter, and tangential velocity parameter (eq. (46)).

FIGURE 9.—Conditions for choked first stator, last rotor, and exhaust nozzle.

Equations (46) and (47) are plotted in figure 9. The blade speed parameter $U/a'_{cr,1}$ is in close agreement with equivalent blade speed $U/\sqrt{\theta'_1}$ if the blade speed parameter is based on the entrance value of critical velocity $a'_{cr,1}$ rather than on $a'_{cr,9}$. The tangential velocity parameter $V_{u,9}/a'_{cr,1}$ is also based on the entrance value of critical velocity $a'_{cr,1}$ in order that the kinetic energy $V^2_{u,9}/2gJ$ may more easily be related to the turbine work.

The use of figure 9 is illustrated by the following example: Consider the operation of a turbine with a rotor-stator area ratio A_8/A_2 of 2.0, a nozzle-stator area ratio A_{10}/A_2 of 2.3, and a blade speed parameter $U/a'_{cr,1}$ of 0.472; for these conditions, the tangential velocity parameter $V_{u,9}/a'_{cr,1}$ is 0. If the nozzle-stator area ratio A_{10}/A_2 is increased from 2.3 to 2.68, the rotor-stator area ratio A_8/A_2 and blade speed parameter $U/a'_{cr,1}$ being held constant, the tangential velocity parameter $V_{u,9}/a'_{cr,1}$ decreases from 0 to −1.0. If the tangential velocity parameter $V_{u,9}/a'_{cr,1}$ is to be increased to 0, the blade speed parameter $U/a'_{cr,1}$ must then be increased from 0.472 to 0.677.

In order to illustrate the effect of adjusting the throat area A_2 of a choked turbine stator on turbine exit conditions, figure 9 includes lines of constant nozzle-rotor area ratio A_{10}/A_8. If the area of only the first turbine stator is changed, the turbine operation will be along a line of constant nozzle-rotor area ratio A_{10}/A_8. If the areas of the first stator and the exhaust nozzle are varied simultaneously, the combined effect of such a variation on the temperature-drop ratio $(T''_8-T'_9)/T'_1$ can be determined by changing both the rotor-stator area ratio A_8/A_2 in inverse proportion to the change in stator area and the nozzle-rotor area ratio A_{10}/A_8 in direct proportion to the change in nozzle area. If the concomitant change in blade speed parameter $U/a'_{cr,1}$ is also known from required conditions of engine operation, the effect of this variation on the tangential velocity parameter $V_{u,9}/a'_{cr,1}$ can be determined.

Recapitulation.—The over-all temperature ratio T'_1/T'_9 across the turbine is shown to depend on only the nozzle-stator area ratio A_{10}/A_2, and it is thus independent of the blade speed parameter $U/a'_{cr,1}$ or the blade profile design. Figure 9 permits variation in the turbine-exit conditions to be approximately estimated for changes in blade speed parameter $U/a'_{cr,1}$ and adjustment of the turbine stator and exhaust-nozzle areas.

NONCHOKED STATOR FOLLOWED BY CHOKED ROTOR

Stator exit tangential velocity parameter.—For choking in the rotor and absence of choking in the stator, equation (8) rather than equation (6) is appropriate. For subsonic flow at the stator exit, the assumptions were made that

$$\left.\begin{aligned} V_2&=V_3 \\ \frac{d\alpha_3}{dV_{u,3}}&=0 \end{aligned}\right\} \tag{48}$$

For a perfect gas,

$$\frac{\rho_2 V_2}{\rho'_1 a'_{cr,1}}=\left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_2}{a'_{cr,1}}\right)^2\right]^{\frac{1}{\gamma-1}}\frac{V_2}{a'_{cr,1}} \tag{49}$$

For subsonic stator exit conditions, equations (8), (12), (48), and (49) were combined to yield

$$\frac{A_2}{A_4}=\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left\{1-\frac{\gamma-1}{\gamma+1}\left(\frac{U}{a'_{cr,1}}\right)\left[2\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right]\right\}^{\frac{\gamma+1}{2(\gamma-1)}}}{\frac{V_{u,3}}{a'_{cr,1}}\sec\alpha_3\left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]^{\frac{1}{\gamma-1}}} \tag{50}$$

which is plotted in figure 10 for two values of stator exit flow angle α_3. At the stator choking limit,

$$\frac{V_3}{a'_{cr,1}}=\frac{V_{u,3}}{a'_{cr,1}}\sec\alpha_3=1$$

Under this condition equation (50) reduces to equation (17).

Differentiation of equation (50) and inclusion of equations (16) and (24) yield

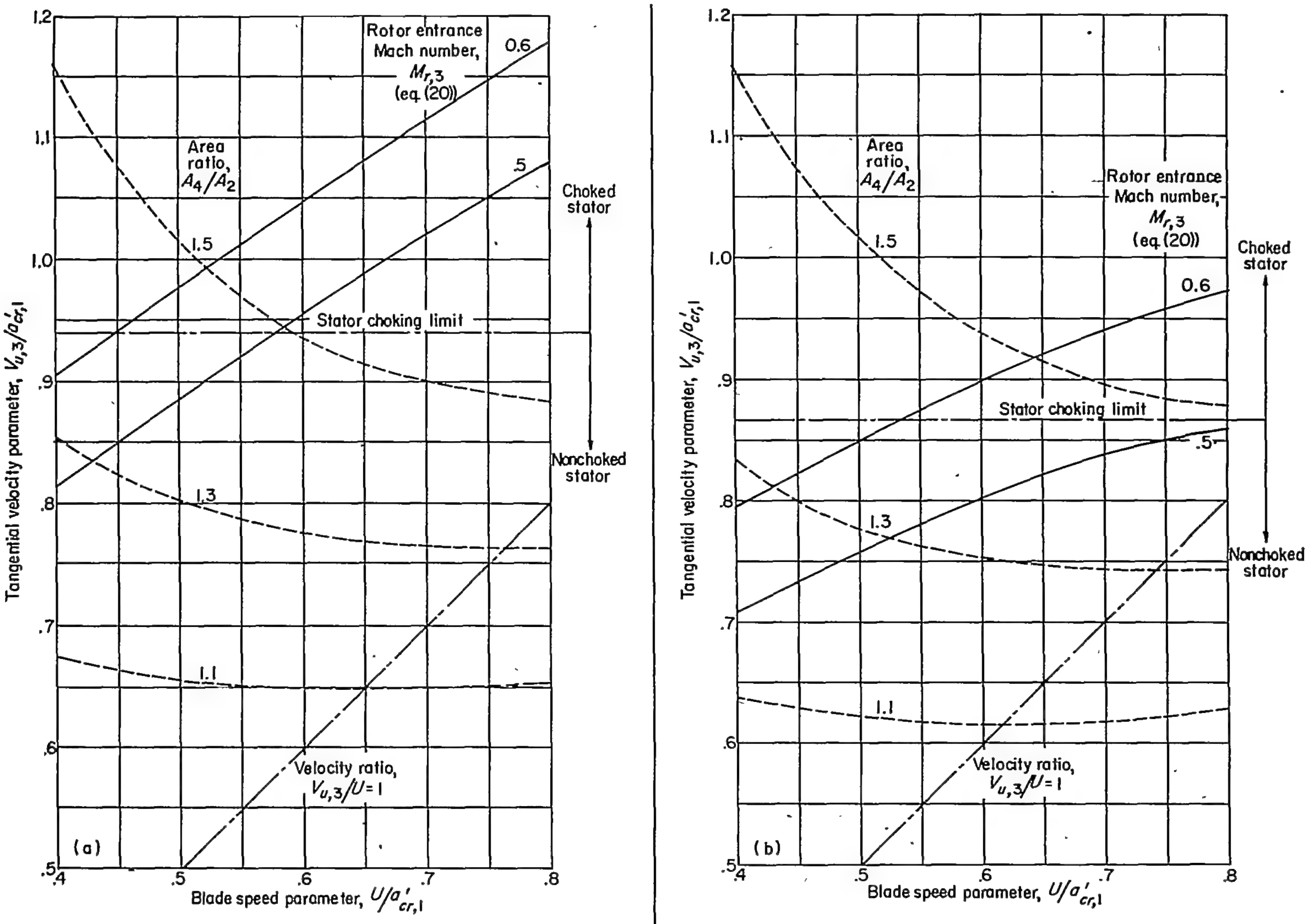
$$\frac{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}{\partial\left(\frac{U}{a'_{cr,1}}\right)}=\frac{1-\frac{V_{u,3}}{U}}{1+\frac{(a'_{cr,1})^2}{UV_{u,3}}\left[1-\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]\left(\frac{T''_3/T'_1}{T_3/T'_1}\right)} \tag{51}$$

At the stator choking limit, equation (51) reduces to equation (18). Because for the subsonic case the denominator of this expression is always positive,

if	$\frac{V_{u,3}}{U}$	$=1$	>1	<1
then	$\frac{\partial\left(\frac{V_{u,3}}{a'_{cr,1}}\right)}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$	$=0$	<0	>0

(52)

Thus, just as for a choked stator, the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ is a minimum if the velocity ratio $V_{u,3}/U$ is 1.



(a) Stator exit flow angle, 20°.

(b) Stator exit flow angle, 30°.

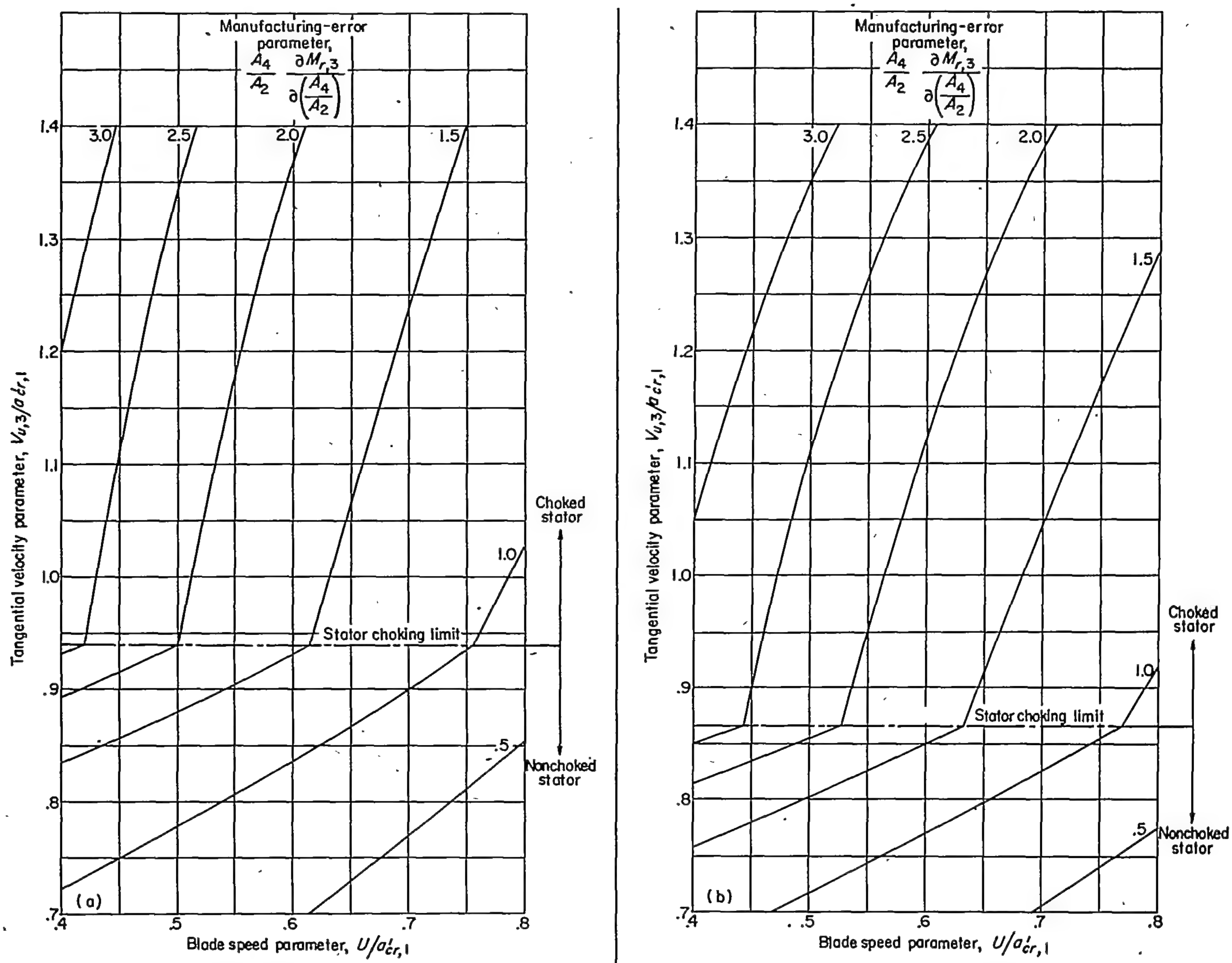
FIGURE 10.—Relation among area ratio, rotor entrance Mach number, and velocity parameters at exit of nonchoked stator followed by choked rotor (eq. (50)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

Effect of manufacturing errors.—In order to make a complete presentation of the effect of manufacturing errors in throat areas upon rotor entrance Mach number for a turbine stage having choking flow in the rotor, the case of a nonchoked stator followed by a choked rotor was analyzed in a manner similar to that used in the derivation of equation (22). Once again, equation (21) was used; evaluation of the four factors on the right side of this identity was made this time by means of equations (20) and (50) to yield the following expression for the manufacturing-error parameter:

$$\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}=\frac{\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\left[1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr,1}}\left(2\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\right]}{\sqrt{1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3}\cdot\sqrt{\frac{\gamma+1}{2}\left[\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\tan^2\alpha_3+\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)^2\right]}}\times$$

$$\frac{\left[\sec^2\alpha_3-\frac{U}{V_{u,3}}-\frac{\gamma-1}{\gamma+1}\sec^2\alpha_3\frac{U}{a'_{cr,1}}\left(\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\right]}{\left\{\frac{UV_{u,3}}{(a'_{cr,1})^2}\left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]+\left[1-\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]\left[1-\frac{\gamma-1}{\gamma+1}\frac{U}{a'_{cr,1}}\left(2\frac{V_{u,3}}{a'_{cr,1}}-\frac{U}{a'_{cr,1}}\right)\right]\right\}} \quad (53)$$

Equation (53) was plotted in figure 11 in the region below the stator choking limit; figure 6 was reproduced in the region above the stator choking limit to complete the presentation. At the stator choking limit (i.e., $V_{u,3}/a'_{cr,1}=\cos\alpha_3$), equations (22) and (53) coincide.



(a) Stator exit flow angle, 20°.

(b) Stator exit flow angle, 30°.

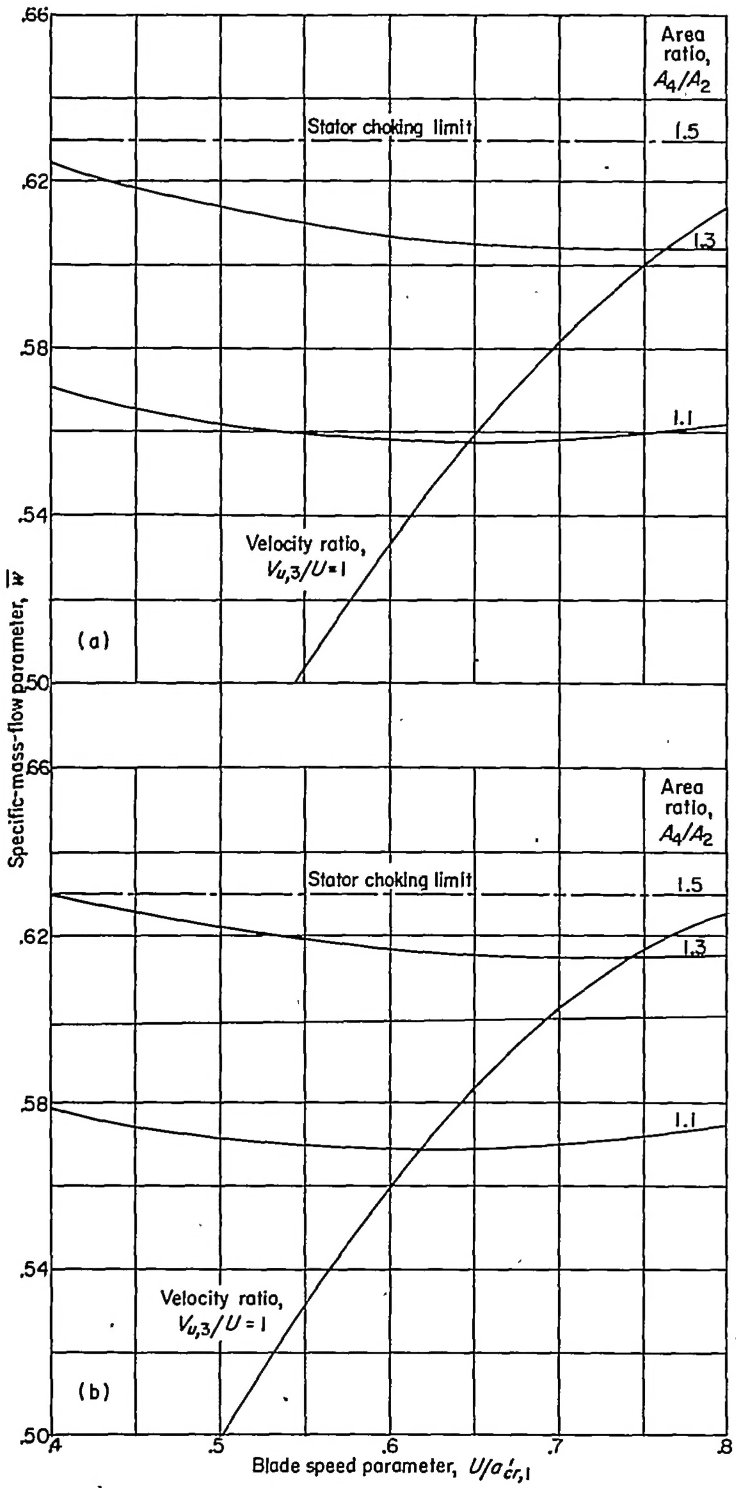
FIGURE 11.—Effect of manufacturing errors in throat areas on rotor entrance Mach number for stator followed by choked rotor. Equation (22) applies above stator choking limit and equation (53) applies below stator choking limit.

The rotor entrance Mach number is frequently, but erroneously, considered to be extremely sensitive to errors in area ratio A_4/A_2 as the stator exit Mach number V_3/a_3 approaches unity. Figure 11 shows that at this stator choking limit, the values of manufacturing-error parameter $\frac{A_4}{A_2}\frac{\partial M_{r,3}}{\partial\left(\frac{A_4}{A_2}\right)}$ remain finite and exhibit no marked variations even in the immediate neighborhood of this limit. It may therefore be concluded that at the choking limit the rotor entrance Mach number is not extremely sensitive to small changes in area ratio A_4/A_2.

Specific-mass-flow parameter.—Equations (8) and (12) were combined to determine the specific-mass-flow parameter $\overline{w}$ where

$$\overline{w}\equiv\frac{\rho_2 V_2}{\rho_1' a_{cr,1}'}=\frac{A_4}{A_2}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left[1-\frac{\gamma-1}{\gamma+1}\frac{U}{a_{cr,1}'}\left(2\frac{V_{u,3}}{a_{cr,1}'}-\frac{U}{a_{cr,1}'}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (54)$$

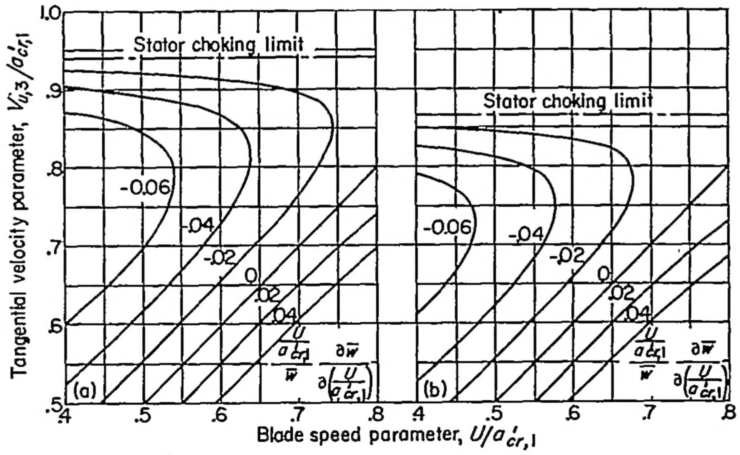
Equation (54) was used in combination with the computations for figure 10 to plot figure 12. For values of area ratio A_4/A_2 sufficiently large to produce choking in the stator, the specific-mass-flow parameter $\overline{w}$ is, of course,



(a) Stator exit flow angle, 20°.
(b) Stator exit flow angle, 30°.

FIGURE 12.—Variation of specific-mass-flow parameter with blade speed parameter and area ratio for nonchoked stator followed by choked rotor (eqs. (50) and (54)).

independent of blade speed parameter $U/a'_{cr,1}$ or area ratio A_4/A_2 (see eq. (4)). For ordinary variations in blade speed parameter $U/a'_{cr,1}$, the variation in specific-mass-flow parameter $\overline{w}$ is comparatively small. The rate of variation in specific-mass-flow parameter $\overline{w}$ with blade speed parameter $U/a'_{cr,1}$ may be determined by differentiating equation (54) and combining this result with equation (51), that is,



(a) Stator exit flow angle, 20°.
(b) Stator exit flow angle, 30°.

FIGURE 13.—Rate of variation of specific-mass-flow parameter with blade speed parameter for nonchoked stator followed by choked rotor (eq. (55)).

$$\frac{\frac{U}{a'_{cr,1}}}{\overline{w}}\frac{\partial\overline{w}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}=\frac{\left(\frac{U}{a'_{cr,1}}\right)^2\frac{T'_1}{T''_3}\left(1-\frac{V_{u,3}}{U}\right)\left[1-\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3\right]}{\left(\frac{T_3}{T''_3}\right)\frac{U\,V_{u,3}}{(a'_{cr,1})^2}+1-\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2\sec^2\alpha_3} \tag{55}$$

Equation (55) is plotted in figure 13 for two values of stator exit flow angle α_3. For the conventional range of design, the factor $\frac{\frac{U}{a'_{cr,1}}}{\overline{w}}\frac{\partial\overline{w}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$ is between 0 and −0.06. From equation (55),

if	$\frac{V_{u,3}}{U}$	$=1$	>1	<1
then	$\frac{\left(\frac{U}{a'_{cr,1}}\right)}{\overline{w}}\frac{\partial\overline{w}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}$	$=0$	<0	>0

(56)

or at the stator choking limit

and thus

$$\left.\begin{aligned}\frac{V_{u,3}}{a'_{cr,1}}&=\cos\alpha_3\\ \frac{\left(\frac{U}{a'_{cr,1}}\right)}{\overline{w}}\frac{\partial\overline{w}}{\partial\left(\frac{U}{a'_{cr,1}}\right)}&=0\end{aligned}\right\} \tag{57}$$

Recapitulation.—Analysis of conditions between a nonchoked stator and a choked rotor indicates that those factors which control the flow conditions between the stator and the rotor are the same as those for a choked stator followed by a choked rotor, namely, blade speed parameter $U/a'_{cr,1}$, area ratio A_4/A_2, and stator exit flow angle α_3. The sensitivity of rotor entrance Mach number $M_{r,3}$ to manufacturing errors

in area ratio A_4/A_2 was determined; rotor entrance Mach number $M_{r,3}$ was found to be only moderately sensitive to errors in area ratio A_4/A_2, even near the stator choking limit. The variation in specific-mass-flow parameter $\overline{w}$ with blade speed parameter $U/a'_{cr,1}$ was determined.

TEST-DATA ANALYSIS FOR A TWO-STAGE TURBINE

The analysis of turbines having choked or nonchoked stators provides an understanding of flow processes in choked-flow turbines which permits some general observations to be made for turbines of this class. In particular, criteria can be set down which will aid in establishing from test data of single-stage or multistage turbines which blade rows are choked and which are not. The following analysis considers a two-stage turbine such as that in figure 1, but such an analysis may be extended to apply to any number of turbine stages.

For a given value of ratio of specific heats γ, equivalent weight flow $w\sqrt{\theta'_1}/\delta'_1$ for any particular turbine is directly proportional to specific-mass-flow parameter $\overline{w}$. Similarly, equivalent blade speed $U/\sqrt{\theta'_1}$ is proportional to blade speed parameter $U/a'_{cr,1}$. The results of the preceding section can thus be employed to predict the manner in which the equivalent weight flow of a given type of turbine will vary with turbine operating conditions.

From the test data, equivalent weight flow at the entrance to each stage, namely, $w\sqrt{\theta'_1}/\delta'_1$ and $w\sqrt{\theta'_5}/\delta'_5$, should be plotted against either the stagnation-pressure ratio p'_1/p'_9 or the stagnation-static-pressure ratio p'_1/p_9 with lines of constant equivalent blade speed $U/\sqrt{\theta'_1}$. In the discussion which follows, it will be assumed that the stagnation-static-pressure ratio p'_1/p_9 is employed. These plots should then be examined in order to determine whether one or more of the following conditions are satisfied:

(1) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}>0$, the flow is not choked. If the pressure ratio can be made sufficiently high, at least one blade row in the turbine will choke. For all higher pressure ratios,

$$\frac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0.$$

(2) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}>0$, the flow is not choked downstream of station 5, that is, in either the second stator or the second rotor. If the pressure ratio can be made sufficiently high, choking will occur at some station downstream of station 5. For all higher pressure ratios, $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$. Under this condition, the flow may or may not be choked upstream of station 5.

(3) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$ but $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{U}{\sqrt{\theta'_1}}\right)}\neq 0$, the first stator is not choked. If, in addition, $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}>0$, then the first rotor is the only blade row which is choked.

(4) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$ and $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{U}{\sqrt{\theta'_1}}\right)}=0$ and at the mean radius $V_{u,3}/U\neq 1$, then the first stator is choked.

(5) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$ but $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{U}{\sqrt{\theta'_1}}\right)}\neq 0$, the second stator is not choked and the second rotor is choked. The preceding blade rows may or may not be choked.

(6) If $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$ and $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_5}}{\delta'_5}\right)}{\partial\left(\dfrac{U}{\sqrt{\theta'_1}}\right)}=0$ and at the mean radius $V_{u,7}/U\neq 1$, then the second stator is choked.

Recapitulation.—These six criteria will in some cases permit discrimination between choking and nonchoking blade rows, but whether or not a particular blade row is choking cannot always be determined. For example, if $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{p'_1}{p_9}\right)}=0$ and $\dfrac{\partial\left(\dfrac{w\sqrt{\theta'_1}}{\delta'_1}\right)}{\partial\left(\dfrac{U}{\sqrt{\theta'_1}}\right)}=0$ and the stator exit velocity ratio $V_{u,3}/U$ at the mean radius is nearly equal to 1, then whether or not the first stator is choking cannot be determined from this test (see eq. (56)).

Even if the first stator is not choked, variations in equivalent weight flow with blade speed may be difficult to detect. Figures 12 and 13 indicate for a nonchoked stator followed by a choked rotor the amount which the specific-mass-flow parameter $\overline{w}$, and thus equivalent weight flow $w\sqrt{\theta'_1}/\delta'_1$, will vary with blade speed parameter $U/a'_{cr,1}$ or, alternatively, equivalent blade speed $U/\sqrt{\theta'_1}$ for constant amount of loss in the turbine. Whether or not the variation in equivalent weight flow can be detected will depend on the range of blade speed for which the rotor is choked and on the precision of measurement.

Any static-pressure measurement within the turbine can also be used to indicate whether or not the flow is choked downstream of the measuring station. For example, if $\frac{\partial(p_1'/p_3)}{\partial(p_1'/p_9)}=0$, then one of the throats downstream of station 3 is choked. If for this condition the second rotor is not choked, this nonchoking can be recognized because $\frac{\partial(p_1'/p_7)}{\partial(p_1'/p_9)}\neq 0$.

TURBINE STATOR ADJUSTMENT

An analysis is needed in order to evaluate the effects of turbine stator adjustment on flow conditions within turbines. Such an analysis, which provides a means for determining the range of usefulness of turbine stator adjustment within aerodynamic limits on internal flow conditions, is presented in the following paragraphs.

Stator adjustment provides an extension of the range of effective turbine operation. When stator adjustment is employed for this purpose, the relation among the turbine operating parameters and the stator setting will depend on the specific application of stator adjustment to be considered. The analysis presented is therefore based on only one application of turbine stator adjustment, and the results are thus useful in appraising turbine stator adjustment for only this application. This general method of analysis may, of course, be employed to study turbine stator adjustment for other applications.

Type of application.—For some applications of turbojet engines, operation of the engine at constant compressor pressure ratio and constant equivalent rotative speed may be desirable. In comparison with the take-off value of engine temperature ratio (turbine inlet temperature to compressor inlet temperature), low values of engine temperature ratio may result from reduced turbine inlet temperature or, for flight at high flight Mach numbers, from increased compressor inlet stagnation temperature. A turbojet engine is therefore considered to be operated in the following manner:

(1) The engine equivalent rotative speed based on compressor inlet conditions is constant.

(2) The compressor pressure ratio is constant.

(3) The engine temperature ratio is varied.

Analysis.—Operating an engine in this way will require adjustment of both the turbine stator and the exhaust nozzle; the equivalent weight flow of the compressor will remain constant. These conditions can be expressed in equation form

$$\frac{w\sqrt{T_c'}}{p_c'}=k \tag{58}$$

$$\frac{U}{\sqrt{T_c'}}=l \tag{59}$$

$$\frac{p_1'}{p_c'}=m \tag{60}$$

where k, l, and m are constants. Combining equations (58) to (60) gives

$$\left(\frac{w\sqrt{T_1'}}{p_1'}\right)\left(\frac{U}{\sqrt{T_1'}}\right)=\frac{kl}{m}$$

or, neglecting variations in γ with engine operating conditions,

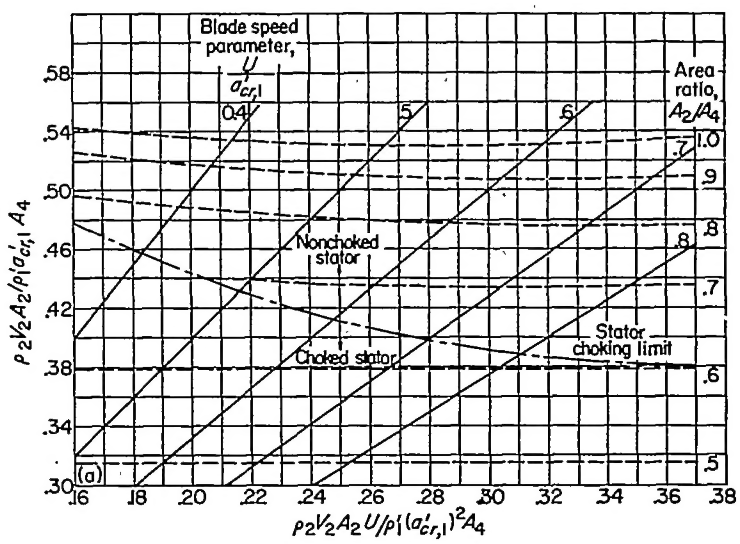
$$\left(\frac{\rho_2 V_2}{\rho_1' a_{cr,1}'}\right)\left(\frac{A_2}{A_4}\right)\left(\frac{U}{a_{cr,1}'}\right)=n=\text{a constant} \tag{61}$$

For operation under the specified conditions, the square of the blade speed parameter $(U/a_{cr,1}')^2$ is inversely proportional to the engine temperature ratio T_1'/T_c'.

Discussion.—For choking of the turbine stator, the specific-mass-flow parameter $\rho_2V_2/\rho_1'a_{cr,1}'$ is a constant. The required variation in area ratio A_2/A_4 is thus proportional to the variation in the reciprocal of blade speed parameter $U/a_{cr,1}'$.

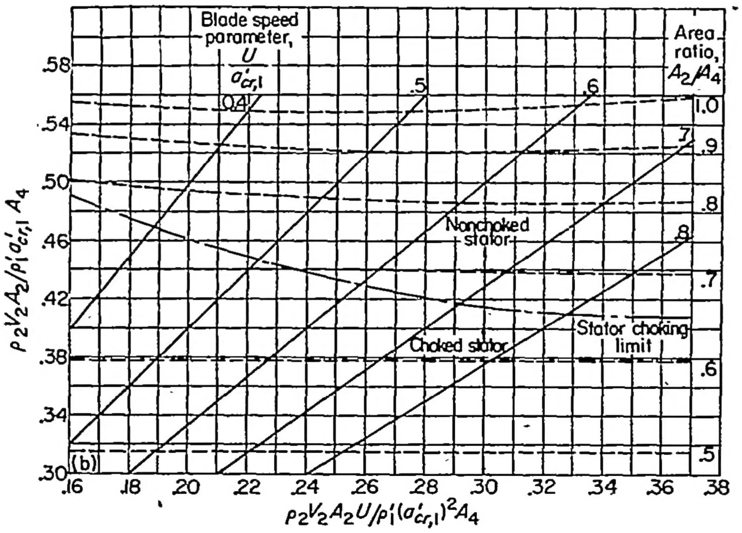
For conditions other than those for which the turbine stator chokes, the variation in specific-mass-flow parameter $\rho_2V_2/\rho_1'a_{cr,1}'$ must be taken into account. For the purposes of this analysis, the turbine rotor was assumed to be choked. A critical stator exit flow angle $\alpha_{cr,3}$ was defined as the maximum value of stator exit flow angle α_3 for which the stator is choked. Then equations (8), (12), (49), and (61) were combined to plot figure 14 for two values of critical stator exit flow angle $\alpha_{cr,3}$.

In figure 14, the lines of constant blade speed parameter $U/a_{cr,1}'$ are straight lines through the origin with slopes equal to $a_{cr,1}'/U$. In the region of stator choking, which is below the stator choking limit line in figure 14, the lines of constant area ratio A_2/A_4 are independent of critical stator exit flow angle $\alpha_{cr,3}$. A comparison of figures 4, 10, and 14 indicates that the more critical rotor entrance Mach numbers occur at high values of blade speed parameter $U/a_{cr,1}'$, which cor-



(a) Critical stator exit flow angle, 15°.

FIGURE 14.—Variation in stator exit flow conditions with stator setting for constant values of compressor equivalent blade speed and compressor pressure ratio.



(b) Critical stator exit flow angle, 25°.

FIGURE 14.—Concluded. Variation in stator exit flow conditions with stator setting for constant values of compressor equivalent blade speed and compressor pressure ratio.

respond to low engine temperature ratio. The variation in rotor entrance conditions with stator setting can be determined from plots such as those in figures 4, 10, and 14.

EFFECT OF VARIATION IN ENTROPY

Although for this analysis the flow through turbines was assumed to be isentropic, the analysis is readily adaptable to include those cases having variations in entropy s. For a perfect gas which undergoes an entropy change, equation (5) becomes

$$e^{\frac{J(s_4-s_2)}{R}}\frac{\rho''_4 a''_{cr,4}}{\rho'_2 a'_{cr,2}}=\left(\frac{T''_4}{T'_2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{62}$$

The area ratio A_2/A_4 in equation (6) is thereby changed to

$$\frac{A_2}{A_4}=\left(\frac{T''_3}{T'_1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}e^{\frac{J(s_2-s_4)}{R}} \tag{63}$$

that is, the change in area ratio is proportional to the factor $e^{J\Delta s/R}$, where Δs is the entropy change between the two stations being considered.

Values of $\overline{w}$ or $\rho_2 V_2/\rho'_1 a'_{cr,1}$ taken from the figures or used in the equations must be multiplied by the factor $e^{J(s_1-s_2)/R}$ to obtain the specific-mass-flow parameter. Because the stagnation temperature relative to a blade row remains constant irrespective of any increase in entropy across the blade row, no adjustment in any of the velocity parameters is necessary. Except for the adjustments in area ratio and specific-mass-flow parameter, the remainder of the analysis presented herein is unchanged.

CONCLUDING REMARKS

Turbines having choking flow were investigated by means of a one-dimensional analysis based on isentropic flow. This analysis indicated that the area ratios, equivalent blade speed, and blade-row exit flow angles are the factors which control design and operation of such turbines. For the usual class of turbine, increasing the equivalent blade speed of a given turbine makes the internal flow conditions become less critical. Flow conditions within choked-flow turbines are not unusually sensitive to manufacturing errors in area ratio even near the stator choking limit. Six criteria are stated that will aid in establishing from test data of turbines which blade rows are choked and which are not. For a turbine equipped with an adjustable stator, the variation in internal flow conditions with operating conditions is determined for one application of stator adjustment. A method is described for extending the analysis to include flow having variations in entropy.

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